

Genuine Correlations of Tripartite System

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We define genuine total, classical and quantum correlations in tripartite systems. The measure we propose is based on the idea that genuine tripartite correlation exists if and only if the correlation between any bipartition does not vanish. We find in a symmetrical tripartite state, for total correlation and classical correlation, the genuine tripartite correlations are no less than pair-wise correlations. However, the genuine quantum tripartite correlation can be surpassed by the pair-wise quantum correlations. Analytical expressions for genuine tripartite correlations are obtained for pure states and rank-2 symmetrical states. The genuine correlations in both entangled and separable states are calculated.

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I. INTRODUCTION

It is believed that various quantum correlations are resources in quantum information processing. Such quantum correlations include entanglement and quantum discord, they are closely related but different from each other. The study of entanglement has a long history [1] but continuously with some new applications [2]. Quantum discord, introduced in Refs. [3–5], also has recently drawn much attention. There is nonzero quantum discord in general separable states in comparison the entanglement vanishes in this case. Additionally, entanglement may disappear at a finite time [6], while quantum discord decays asymptotically even at a finite temperature, which is known as the robustness of quantum discord [7]. In general it seems that quantum discord is more common than entanglement in quantum states. For specific applications of quantum discord in quantum information protocols, the deterministic quantum computation with one qubit (DQC1) [8] and state discrimination [9, 10] can be realized using states with zero entanglement but positive quantum correlation. The importance of quantum discord is also presented in quantum communication [11]. Generally, numerical methods is required for calculating quantum discord. By far, analytical evaluation of quantum discord are developed for some special two-qubit states, e. g. X states [12, 13]. A simple method is pointed out to get the analytical quantum discord of a rank-two bipartite state by relationship with entanglement of formation [14].

Despite the great potential advantages of multipartite quantum correlations, the problem of quantifying and characterizing the correlation contained in the multipartite quantum systems is still open. Recently, much effort has been devoted in measuring the quantum correlations in multipartite systems. A global measure is proposed for quantum correlations of multipartite systems [15]. The quantum dissension which measures the quantum correlations in a tripartite state

is introduced in Ref. [16]. Moreover in Ref. [17], the genuine total, quantum and classical correlations in multipartite system are defined by employing relative entropy as a distance measure of correlations [18]. Some general criteria about the definition of genuine n -partite correlations has been given by Bennett *et al.* in Ref. [19].

In this paper, we try to provide some new definitions of genuine tripartite correlations. The genuine total or quantum correlation goes to zero if and only if there is a bipartition of the tripartite system such that no total or quantum correlation exist between the two parts. We obtain the analytical expressions for the genuine tripartite correlations in pure and rank-two states. Genuine total, quantum and classical correlations are compared with their pairwise counterparts. We find that the genuine total and classical correlations are greater than their pairwise counterparts in symmetrical states, but this quantitative relation does not hold for quantum correlations. Our definition of genuine tripartite quantum correlation is no more than the one defined in [17], but the two definitions coincide for pure states.

The remainder of this paper is arranged as follows. In Sec II, we give our new definitions of genuine tripartite correlations and investigate the case of pure and symmetrical tripartite states. In Sec III, we will illustrate by examples the tripartite genuine quantum discord and classical correlations in symmetrical systems. Sec IV is devoted to the conclusion.

II. AN OPTIMAL GENUINE TRIPARTITE CORRELATIONS

Let us briefly recall the definition of two-particle quantum discord [3]:

$$D(\rho_{a:b}) \equiv I(\rho_{a,b}) - J(\rho_{a:b}), \quad (1)$$

where the mutual information $I(\rho_{a,b}) = S(\rho_a) + S(\rho_b) - S(\rho_{ab})$, with $S(\rho) = -\text{tr}(\rho \log_2 \rho)$, characterizes the total correlations, and $J(\rho_{a:b}) = S(\rho_a) - \min_{\{E_i^b\}} S(\rho_{a|\{E_i^b\}})$ is called the classical correlation. Here $\{E_i^b\}$ is positive operator valued measures (POVM) performed on system b and the conditional entropy

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is $S(\rho_{a|E_l^b}) = \sum_l p_l S(\rho_{a|E_l^b})$, where $\rho_{a|E_l^b} = \text{Tr}_b(E_l^b \rho_{ab})/p_l$ and $p_l = \text{Tr}_{ab}(E_l^b \rho_{ab})$.

A state of n particles is said to possess genuine n -partite correlations when it is nonproduct in every bipartite cut, according to [19]. From this point of view, we can define genuine tripartite correlations in tripartite states $\rho_{abc} \equiv \rho$ as

$$T^{(3)}(\rho) \equiv \min[I(\rho_{a,bc}), I(\rho_{b,ac}), I(\rho_{c,ab})], \quad (2)$$

where $I(\rho_{i,jk}) = S(\rho_i) + S(\rho_{jk}) - S(\rho)$ is the mutual information between one-qubit part and the left two-qubit part, which goes to zero if and only if $\rho = \rho_i \otimes \rho_{jk}$. This definition of genuine total correlation coincides with the one defined in Ref. [17].

Consistently with the definition of $T^{(3)}$, a tripartite state does not have genuine tripartite quantum discord when there exist a bipartition such that the quantum correlation between the two parts is zero. Obviously, $D(\rho_{i,jk})$ equals zero if $D(\rho_{ij;k})$ takes zero, but the converse is not true. Therefore, we define genuine tripartite quantum discord as:

$$D^{(3)}(\rho) = \min[D(\rho_{a,bc}), D(\rho_{b,ac}), D(\rho_{c,ab})], \quad (3)$$

where $D(\rho_{i,jk}) = S(\rho_{jk}) + S(\rho_{i|jk}) - S(\rho)$, $S(\rho_{i|jk}) = \min_{\{E_m^{ij}\}} [S(\rho_{k|E_m^{ij}})]$, and $\{E_m^{ij}\}$ is a two-particle POVM operating on i and j . Then, we define genuine tripartite classical correlations as:

$$J^{(3)}(\rho) = T^{(3)}(\rho) - D^{(3)}(\rho). \quad (4)$$

Without loss of generality, we make the following assumption:

$$I(\rho_{c,ab}) \leq I(\rho_{a,bc}) \leq I(\rho_{b,ac}). \quad (5)$$

The left discussion in this section is based on it. For pure states, the relative entropy $S(\rho_{c|ab}) = 0$. By assumption (5), we have $S(\rho_c) = S(\rho_{ab}) \leq S(\rho_a) = S(\rho_{bc}) \leq S(\rho_b) = S(\rho_{ac})$. Therefore, we can obtain the genuine tripartite total, classical correlations and quantum discord:

$$\begin{aligned} T^{(3)}(\rho) &= 2S(\rho_c) = 2S(\rho_{ab}), \\ D^{(3)}(\rho) &= J^{(3)}(\rho) = S(\rho_{ab}) = S(\rho_c). \end{aligned} \quad (6)$$

It means that genuine tripartite classical correlations and quantum discord are both equal to half of the genuine tripartite total correlations in a pure state, which coincide with the genuine correlations defined in Ref [17].

When the tripartite quantum system is symmetrical, i. e., the state of the whole system is invariant under the permutations of the three parties, the genuine tripartite total, classical and quantum correlations can be regard as:

$$\begin{aligned} T^{(3)}(\rho) &= S(\rho_c) + S(\rho_{ab}) - S(\rho), \\ J^{(3)}(\rho) &= S(\rho_c) - S(\rho_{c|ab}), \\ D^{(3)}(\rho) &= S(\rho_{ab}) + S(\rho_{c|ab}) - S(\rho). \end{aligned} \quad (7)$$

We derive some properties of the genuine correlations.

Theorem. For a symmetrical tripartite quantum state, the genuine tripartite total and classical correlations is no less than the any pairwise counterpart, respectively

$$\begin{aligned} T^{(3)}(\rho) &\geq T^{(2)}(\rho), \\ J^{(3)}(\rho) &\geq J^{(2)}(\rho), \end{aligned} \quad (8)$$

where the $T^{(2)}$ and $J^{(2)}$ are pairwise total and classical correlations respectively.

Proof of Theorem –The mutual information does not increase when discard quantum subsystem: $I(\rho_{c,a}) \leq I(\rho_{c,ab})$ [20], it is obvious that genuine tripartite correlations is no less than pairwise correlations of symmetrical tripartite systems. For classical correlations, we have $J^{(3)}(\rho) = J(\rho_{c,ab})$ and $J^{(2)}(\rho) = J(\rho_{c,a})$. Direct calculations lead to $J^{(3)}(\rho) - J^{(2)}(\rho) = S(\rho_{c|a}) - S(\rho_{c|ab})$. Notice that $S(\rho_{c|a}) = \min_{\{E^a \otimes I^b\}} S(\rho_{c|E^a \otimes I^b})$, and that $\{E_l^a \otimes I^b\}$ may not be the optimal POVM $\{E_l^{ab}\}$ in the definition of $S(\rho_{c|ab})$. Therefore, we have $J^{(3)}(\rho) \geq J^{(2)}(\rho)$. This completes the proof.

For quantum correlations, there are no fixed quantitative relation between genuine and pairwise quantum correlations, which we will illustrate in the next section by some concrete examples.

III. ANALYTIC EXPRESSION OF GENUINE TRIPARTITE QUANTUM DISCORD FOR RANK-TWO SYMMETRICAL STATES

We now consider genuine quantum discord of rank-two symmetrical states of three qubits, which we can get the analytic results. A rank-two symmetrical tripartite system can be written as

$$\rho = p|\varphi_1\rangle\langle\varphi_1| + (1-p)|\varphi_2\rangle\langle\varphi_2|, \quad (9)$$

where $|\varphi_i\rangle$ is a three-qubit symmetrical state. The state as in Eq. (9) can be purified to a four-qubit pure state by attaching an auxiliary system d :

$$|\Psi_{abcd}\rangle = \sqrt{p}|\varphi_1, 0\rangle + \sqrt{1-p}|\varphi_2, 1\rangle. \quad (10)$$

Then the Koashi-Winter relation [14] gives

$$D^{(3)}(\rho) = S(\rho_{ab}) + E(\rho_{cd}) - S(\rho). \quad (11)$$

Here $E(\rho_{cd})$ is the entanglement of formation (EOF) between qubits c and d , which is defined as

$$E(\rho_{cd}) = \min_{\{p_i, |\phi_i\rangle_{cd}\}} \sum_i p_i S(\text{Tr}_c(|\phi_i\rangle\langle\phi_i|)), \quad (12)$$

and can be calculated as follows. $E(\rho_{cd}) = -h \log_2 h - (1-h) \log_2 (1-h)$, where $h = \frac{1+\sqrt{1-C_{cd}^2}}{2}$, C_{cd} being the concurrence of ρ_{cd} [22, 23]. Optimal POVM $\{E_i^{ab}\}$ in the definition of $D^{(3)}$ related to the optimal pure state decomposition $\{p_i, |\phi_i\rangle_{cd}\}$ for EOF as follows [14]:

$$\rho_{c|ab}^i = \text{Tr}_{ab}\{E_i^{ab}\rho\}/p_i = \text{Tr}_d\{|\phi_i\rangle\langle\phi_i|\}. \quad (13)$$

We study two concrete examples to investigate more closely the properties of genuine tripartite correlations. Firstly, consider a symmetric tripartite system as the form:

$$\rho = p|000\rangle\langle 000| + (1-p)|\phi\phi\phi\rangle\langle\phi\phi\phi| \quad (14)$$

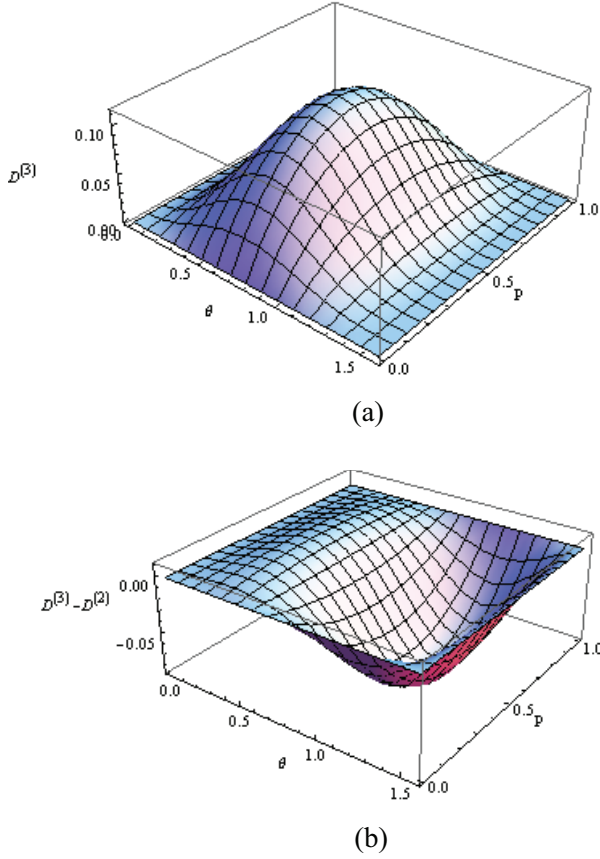


FIG. 1: (a): $D^{(3)}(\rho)$ as a function of θ and p . (b): $D^{(3)}(\rho) - D^{(2)}(\rho)$ as function of θ and p .

where $|\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$. This is a three-qubit product state with no entanglement of any type. We calculate nonzero eigenvalues of ρ and ρ_{ab} as well as the analytical formula of h

$$\begin{aligned}
 \lambda_1^{abc} &= \frac{1}{8}(4 - \sqrt{2}\sqrt{8 - pq(22 - 15\cos 2\theta - 6\cos 4\theta - \cos 6\theta)}), \\
 \lambda_2^{abc} &= \frac{1}{8}(4 + \sqrt{2}\sqrt{8 - pq(22 - 15\cos 2\theta - 6\cos 4\theta - \cos 6\theta)}), \\
 \lambda_1^{ab} &= \frac{1}{4}(2 - \sqrt{2}\sqrt{2 - pq(5 - 4\cos 2\theta - \cos 4\theta)}), \\
 \lambda_2^{ab} &= \frac{1}{4}(2 + \sqrt{2}\sqrt{2 - pq(5 - 4\cos 2\theta - \cos 4\theta)}), \\
 h &= \frac{1}{8}(4 + (16 + 15pq(\cos 2\theta - 18 + 2\cos 4\theta + \cos 6\theta) \\
 &\quad + 8\sqrt{pq(\sin\theta)^6\sqrt{pq(5\sin\theta + \sin 3\theta)^2}})^{\frac{1}{2}}), \quad (15)
 \end{aligned}$$

where $q = 1 - p$. Then from Eq. (11), we obtain the analytic expression for the genuine tripartite quantum discord:

$$\begin{aligned}
 D^{(3)}(\rho) &= -\lambda_1^{ab}\log_2\lambda_1^{ab} - \lambda_2^{ab}\log_2\lambda_2^{ab} - (1-h)\log_2(1-h) \\
 &\quad - h\log_2 h + \lambda_1^{abc}\log_2\lambda_1^{abc} + \lambda_2^{abc}\log_2\lambda_2^{abc}. \quad (16)
 \end{aligned}$$

The analytical results are plotted in Fig. 1. Figure (1a) is $D^{(3)}(\rho)$ as a function of p and θ . We see that the $D^{(3)}(\rho)$

equals to zero when $p = 0$ or $p = 1$ and it takes the maximal value for $p = \frac{1}{2}$, when θ is fixed. It is not difficult to find that $D^{(3)}(\rho)$ is symmetric for p with $p = \frac{1}{2}$ the symmetric axis. This can be understood as follows. The state in Eq. (14) can be transformed into $\rho' = (1-p)|000\rangle\langle 000| + p|\phi\phi\phi\rangle\langle\phi\phi\phi|$ by the unitary operator $U' = U \otimes U \otimes U$, where $U = \{\cos\theta, \sin\theta; \sin\theta, -\cos\theta\}$. Genuine correlations are preserved under local unitary operations, that is $D^{(3)}(\rho) = D^{(3)}(\rho')$. Hence, $D^{(3)}(\rho)$ is invariant when p and $1-p$ are interchanged. Then, we find that $D^{(3)}(\rho)$ takes the maximal value when $p = \frac{1}{2}$ and $\theta = 0.688$. In Fig. 1(b) is plotted the difference between $D^{(3)}$ and $D^{(2)}$ as a function of p and θ . The cases which $D^{(3)}(\rho)$ is less than, equal to or greater than $D^{(2)}(\rho)$ are all possible.

We then turn to find out the corresponding optimal measurements by which we get the genuine tripartite quantum discord. The case for $p = \frac{1}{2}$ is discussed in here. Firstly, we achieve the optimal pure state decomposition of ρ_{cd} which minimized the EOF of the state using the method in Ref. [23]:

$$\begin{aligned}
 |\phi_1\rangle &= \frac{1}{2}(\sqrt{1 + \cos^2\theta} + \sin\theta)|00\rangle \\
 &\quad + \frac{1}{2}\cos\theta(\sqrt{1 + \cos^2\theta} - \sin\theta)|01\rangle \\
 &\quad + \frac{1}{2}(\sqrt{1 + \cos^2\theta} - \sin\theta)\sin\theta|11\rangle, \\
 |\phi_2\rangle &= \frac{1}{2}(\sqrt{1 + \cos^2\theta} - \sin\theta)|00\rangle \\
 &\quad + \frac{1}{2}(\cos\theta\sqrt{1 + \cos^2\theta} + \cos\theta\sin\theta)|01\rangle \\
 &\quad + \frac{1}{2}(\sqrt{1 + \cos^2\theta}\sin\theta + \sin^2\theta)|11\rangle. \quad (17)
 \end{aligned}$$

Then from Eq. (13), we have the optimal measurement bases for $D^{(3)}$ as

$$\begin{aligned}
 |E_1^{ab}\rangle &= \frac{\sqrt{3 + \cos 2\theta} + \sqrt{2}\sin\theta}{2\sqrt{2}}|00\rangle \\
 &\quad - \frac{\sin\theta(\sqrt{3 + \cos 2\theta} - \sqrt{2}\sin\theta)}{2\sqrt{3 + \cos 2\theta}}|11\rangle \\
 &\quad - \frac{\cos\theta(\sqrt{3 + \cos 2\theta} - \sqrt{2}\sin\theta)}{2\sqrt{3 + \cos 2\theta}}(|01\rangle + |10\rangle), \quad (18)
 \end{aligned}$$

$$\begin{aligned}
 |E_2^{ab}\rangle &= \frac{\sqrt{3 + \cos 2\theta} - \sqrt{2}\sin\theta}{2\sqrt{2}}|00\rangle \\
 &\quad + \frac{\sin\theta(\sqrt{3 + \cos 2\theta} + \sqrt{2}\sin\theta)}{2\sqrt{3 + \cos 2\theta}}|11\rangle \\
 &\quad + \frac{\cos\theta(\sqrt{3 + \cos 2\theta} + \sqrt{2}\sin\theta)}{2\sqrt{3 + \cos 2\theta}}(|01\rangle + |10\rangle). \quad (19)
 \end{aligned}$$

The other two measurements with above two measurements constitute a set of orthogonal basis which satisfy that $\sum_{k=1}^4 |E_k^{ab}\rangle\langle E_k^{ab}| = I$. It must be noticed $|E_k^{ab}\rangle$ can not always be written as the form that $|E_l^a\rangle \otimes |E_m^b\rangle$.

The genuine tripartite classical and quantum correlations we defined are different with which defined in [17]. Here

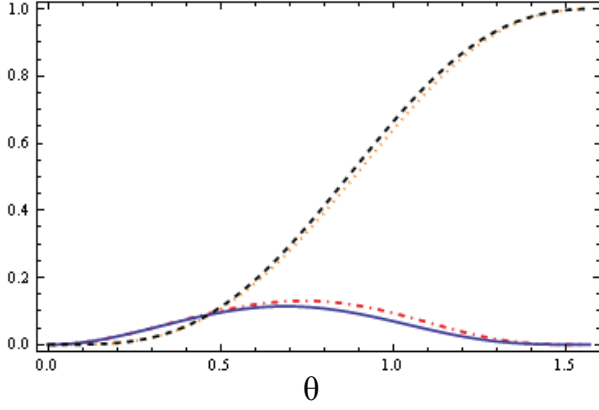


FIG. 2: Correlations for the state, $\rho = p|000\rangle\langle 000| + (1-p)|\phi\phi\phi\rangle\langle\phi\phi\phi|$, vary with θ when $p = 1/2$: $D^{(3)}(\rho)$ (blue solid line), $J^{(3)}(\rho)$ (black dashed line), $D'^{(3)}(\rho)$ (red dot-dashed line), $J'^{(3)}(\rho)$ (orange dotted line).

we compare our measure for genuine correlations with those defined in Ref [17], where $J'^{(3)}(\rho) = S(\rho_c) - S'(\rho_{c|ab})$ and $D'^{(3)}(\rho) = S(\rho_{ab}) + S'(\rho_{c|ab}) - S(\rho_{abc})$ with $S'(\rho_{c|ab}) = \min_{\{E_l^a, E_m^b\}} [S(\rho_{c|\{E_l^a, E_m^b\}})]$ are the genuine quantum and classical correlations. Since $\{E_l^a \otimes E_m^b\}$ may not be the optimal POVM $\{E_m^{ab}\}$ in the definition of $D^{(3)}$, we have $D^{(3)}(\rho) \leq D'^{(3)}(\rho)$ and $J^{(3)}(\rho) \geq J'^{(3)}(\rho)$. The comparison is shown in figure 2 which is the θ -dependent correlations variation curves when $p = \frac{1}{2}$. From figure 2, we can see that $D^{(3)}(\rho)$ and $D'^{(3)}(\rho)$ are quite close to each other. However, the two measures of genuine quantum correlation do not coincide for $\theta \neq 0$ or $\pi/2$. It means that even for separable state in Eq. (14), the optimal measurement $\{E_m^{ab}\}$ in the definition of genuine quantum correlation $D^{(3)}$ can not be written as $E_m^{ab} = E_l^a \otimes E_m^b$. Another interesting phenomenon is that the genuine quantum correlation $D^{(3)}(\rho)$ may surpass the genuine classical correlation $J^{(3)}(\rho)$ even for separable states.

Now, we consider a state of this form

$$\rho = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W|, \quad (20)$$

where $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$. The genuine tripartite discord of the state as in equation (20) can be calculated using the above method

$$\begin{aligned} D^{(3)}(\rho) = & -\left(\frac{1-p}{3} + \frac{p}{2}\right) \log_2\left(\frac{1-p}{3} + \frac{p}{2}\right) \\ & -\left(\frac{2}{3}(1-p)\right) \log_2\left(\frac{2}{3}(1-p)\right) - \left(\frac{p}{2}\right) \log_2\left(\frac{p}{2}\right) \\ & + p \log_2 p + (1-p) \log_2(1-p). \end{aligned} \quad (21)$$

Figure 3(a) shows $D^{(3)}(\rho)$ is greater than $D^{(2)}(\rho)$. There is a transition point at $p = 0.51$ for $D^{(2)}(\rho)$ because of the sudden change of the measurement basis. The three-tangle τ_3 [25] of the state (20) which we showed in the Fig. 3(a) has been obtained in ref [24]. We can see that the genuine quantum discord is no less than the three-tangle τ_3 . Fig. 3(b) shows $D^{(3)}(\rho)$, $D^{(2)}(\rho)$, $J^{(3)}(\rho)$ and $J'^{(3)}(\rho)$ as functions of p . We can

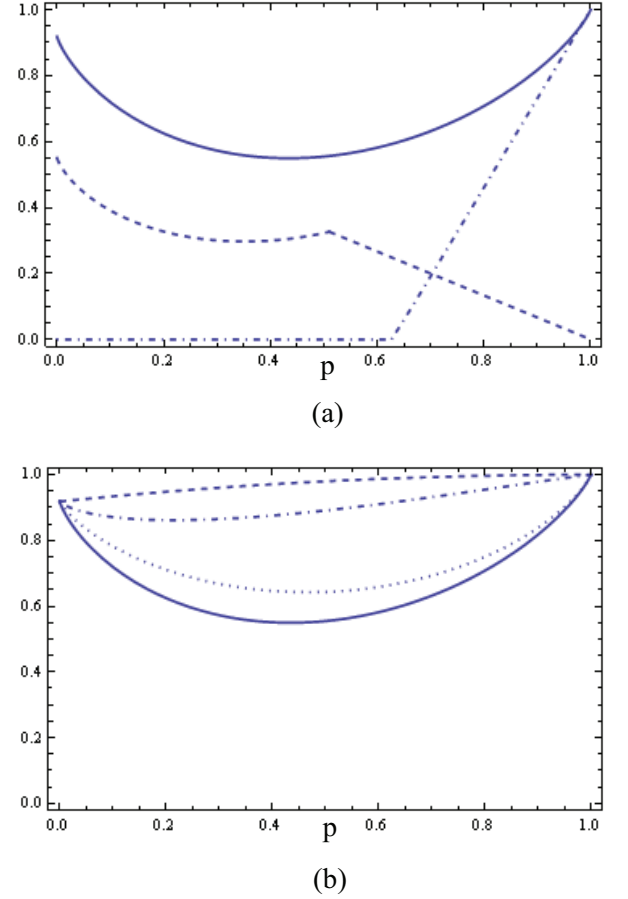


FIG. 3: Correlations for the state, $\rho_{abc} = p|GHZ\rangle\langle GHZ| + (1-p)|W\rangle\langle W|$, vary with p . (a): $D^{(3)}(\rho)$ (solid line), $D^{(2)}(\rho)$ (dashed line), τ_3 (dot-dashed line). (b): $D^{(3)}(\rho)$ (solid line), $J^{(3)}(\rho)$ (dashed line), $D'^{(3)}(\rho)$ (dot-dashed line), $J'^{(3)}(\rho)$ (dotted line).

see that the four quantities coincide only for $p = 0$ or $p = 1$, where the state in Eq. (20) is just the $|W\rangle$ state or $|GHZ\rangle$ state. For $p \in (0, 1)$, we can see that in this state, the gap between $D'^{(3)}(\rho)$ and $D^{(3)}(\rho)$, as well as that between $J^{(3)}(\rho)$ and $J'^{(3)}(\rho)$ can be very large. Moreover, the ordering is different, i.e., $J^{(3)}$ is greater than $D^{(3)}$ while $J'^{(3)}$ is less than $D'^{(3)}$.

IV. CONCLUSION

In summary, we have investigated the genuine correlations in a tripartite quantum state. We propose the definitions for genuine tripartite quantum and classical correlations, as well as the analytical expression of them for rank-two symmetrical states of three qubits. We have shown that, genuine tripartite classical correlations and quantum discord are both equal to half of genuine tripartite total correlations in pure tripartite states, which coincide with the definition of genuine correlation given in [17]. For a symmetrical tripartite state, the quantitative relation between genuine tripartite quantum discord and its pairwise counterpart is not fixed, while the genuine tripartite total and classical correlations are no less than their

any pairwise counterparts. Interestingly, the genuine quantum correlation can surpass the genuine classical correlation even in some separable states.

The study of various correlations in multipartite states is of interests not only for quantum information science but also for many-body systems in condensed matter physics and statistical mechanics. However, no consensual measures of various correlations in multipartite case are found, even in the

well-studied entanglement case. The correlation measures in tripartite states proposed in this paper should be a start point in completely quantifying the multipartite correlations. It will also be interesting to use the correlation measures presented in this paper in some real physical systems.

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